

# The Diophantine Equation

$$\arctan \left( \frac{1}{\mathbf{x}} \right) + \arctan \left( \frac{\ell}{\mathbf{y}} \right) = \arctan \left( \frac{1}{\mathbf{k}} \right)$$

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## 1 Introduction

The subject matter of this work is the two-variable diophantine equation  $\arctan \left( \frac{1}{x} \right) + \arctan \left( \frac{\ell}{y} \right) = \arctan \left( \frac{1}{k} \right)$  for given positive integers  $k$  and  $\ell$ , such that  $\gcd(\ell, k^2 + 1) = 1$  (i.e.,  $\ell$  and  $k^2 + 1$  are relatively prime). The main objective is to determine all positive integer pairs  $(x, y)$  which satisfy

$$\left\{ \begin{array}{l} \arctan\left(\frac{1}{x}\right) + \arctan\left(\frac{\ell}{y}\right) = \arctan\left(\frac{1}{k}\right) \\ x, y \in \mathbb{Z}^+, \gcd(\ell, k^2 + 1) = 1 \text{ and} \\ \text{with } \gcd(\ell, y) = 1 \text{ (i.e., } \ell \text{ and } y \text{ are} \\ \text{relatively prime)} \end{array} \right\} \quad (1)$$

This is done in Theorem 1, Section 4. As we will see, there are exactly  $N$  distinct solutions to (1) where  $N$  is the number of positive divisors of the integer  $k^2 + 1$ . The  $N$  pairs  $(x, y)$ , which are solutions to (1), are expressed parametrically in terms of the positive divisors of  $k^2 + 1$ . Also, note that when  $\ell = 1$ , equation (1) is symmetric with respect to the two variables  $x$  and  $y$ . If  $(a, b)$  is a solution, then so is  $(b, a)$ . The motivating force behind this work is a recent article published in the journal *Mathematics and Computer Education* (see [1]). The article, authored by Hasan Unal, is entitled “Proof without words: an arctangent equality”. It consists of four illustrations, a purely geometric proof of the equality,

$$\arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{7}\right) = \arctan\left(\frac{1}{2}\right).$$

From the point of view of (1), the last equality says that the pair  $(3, 7)$  is a solution of (1), in the case  $\ell = 1$  and  $k = 2$ .

According to Theorem 1,  $(3, 7)$  and  $(7, 3)$  are the only solutions to (1) for  $\ell = 1$  and  $k = 2$ .

This, then, is the other objective of this article. To generate more arctangent type of equalities. This is done in Section 5, where a listing of such equalities is offered; an immediate consequence of Theorem 1.

In Section 2, we list two trigonometric preliminaries: the well known identity for the tangent of the sum of two angles and a couple of basic facts regarding arctangent function.

In Section 3, we state two well known results from number theory: Euclid’s lemma; and the formula that gives the number of positive divisors of a positive integer. We use these in the proof of Theorem 1.

## 2 Trigonometric preliminaries

- (a) If  $\theta_1$  and  $\theta_2$  are two angles measured in radians, such that neither  $\theta_1$  nor  $\theta_2$ , nor their sum  $\theta_1 + \theta_2$  is of the form  $k\pi + \frac{\pi}{2}$ ,  $k$  and integer.

Then,

$$\tan(\theta_1 + \theta_2) = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2}$$

(b) Let  $f$  be the arctangent function,  $f(x) = \arctan x$ . Then,

$$\begin{aligned} \text{(i)} \quad & \arctan 1 = \frac{\pi}{4} \\ \text{(ii)} \quad & \left\{ \begin{array}{l} 0 < \theta < \frac{\pi}{r} \\ \text{and} \\ \theta = \arctan c \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} 0 < \theta < \frac{\pi}{4} \\ 0 < c = \tan \theta < 1 \end{array} \right\} \\ \text{(iii)} \quad & \left\{ \begin{array}{l} 0 < \theta < \frac{\pi}{2} \\ \text{and} \\ \theta = \arctan c \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} 0 < \theta < \frac{\pi}{2} \\ 0 < c = \tan \theta \end{array} \right\} \end{aligned}$$

### 3 Number theory preliminaries

The following result is commonly known as Euclid's lemma, and is of great significance in number theory.

**Result 1 (Euclid's lemma):** *Let  $a, b, c$  be positive integers such that  $a$  is a divisor of the product  $bc$ ; and with  $a$  also being relatively prime to  $b$ . Then,  $a$  is a divisor of  $c$ .*

The next result provides a formula that gives the exact number of positive divisors of a positive integer.

**Result 2 (number of divisors formula)** *Let  $n \geq 2$  be a positive integer, and let  $p_1, \dots, p_t$  in increasing order, be the distinct prime bases that appear in the prime factorization of  $n$ , so that  $n = p_1^{e_1} \dots p_t^{e_t}$ , with the exponents  $e_1, \dots, e_t$  being positive integers. Also, let  $N$  be the number of positive divisors of  $n$ . Then,*

$$(i) \quad N = \prod_{i=1}^t (e_i + 1) \dots (e_1 + 1) \dots (e_t + 1).$$

(ii) *In particular, when  $e_1 = \dots = e_t = 1$  (i.e., when  $n$  is squarefree)*

$$N = 2^t$$

Both of these two results can be easily found in number theory books and texts. For example, see reference [2].

## 4 Theorem 1 and its proof

**Theorem 1.** *Let  $k$  and  $\ell$  be fixed or given positive integers such that  $\gcd(\ell, k^2 + 1) = 1$ . Consider the diophantine equation (1).*

- (a) *There are exactly  $N$  distinct positive integer pairs  $(x, y)$  which are solutions to equation (1) where  $N$  is the number of positive integer divisors of the integer  $k^2 + 1$ . Specifically, if  $(x, y)$  is a positive integer solution of (1), then*

$$x = k + \ell \cdot \left( \frac{k^2 + 1}{d} \right) \text{ and } y = k\ell + d \text{ where } d \text{ is a positive integer divisor of } k^2 + 1.$$

- (b) *If  $k^2 + 1 = p$ , a prime number, then equation (1) has exactly two distinct positive integer solutions. These are*

$$(x, y) = (k + \ell(k^2 + 1), k\ell + 1), \quad (k + \ell, k\ell + k^2 + 1).$$

- (c) *If  $k^2 + 1 = p_1 p_2$ , a product of two distinct primes  $p_1$  and  $p_2$ , equation (1) has exactly four distinct positive integer solutions. These are,*

$$(x, y) = (k + \ell(k^2 + 1), k\ell + 1), \quad (k + \ell, k\ell + k^2 + 1),$$

$$(k + \ell p_2, k\ell + p_1), \text{ and } \quad (k + \ell p_1, k\ell + p_2)$$

*Proof.* First note that parts (b) and (c) are immediate consequences of part (a) and Result 2. We omit the details. We prove part (a)

- (a) Let  $d$  be a positive integer divisor of  $k^2 + 1$ . We will show that the positive integer pair,  $(x_d, y_d) = \left( k + \ell \cdot \left( \frac{k^2 + 1}{d} \right), k\ell + d \right)$  is a solution to (1). First note that  $y_d = k\ell + d$ , is relatively prime to  $\ell$ . Indeed, if  $y_d$

and  $\ell$  had a prime factor  $q$  in common then  $q$  would divide  $y_d - k\ell = d$ ; and thus (since  $d$  is a divisor of  $k^2 + 1$ )  $y_d - k\ell = d$ , then  $q$  would divide  $k^2 + 1$  contrary to the hypothesis that  $\gcd(\ell, k^2 + 1) = 1$ . Thus,  $\gcd(\ell, y_d) = 1$ .

It is clear that since  $k, \ell$  and  $d$  are positive integers, we have  $x_d > 1, y_d > 1$  and  $k \geq 1$ . So,

$$\left( 0 < \frac{1}{x_d} < 1, 0 < \frac{\ell}{y_d} < 1, 0 < \frac{1}{k} \leq 1 \right). \quad (2)$$

Let

$$\theta_1 = \arctan\left(\frac{1}{x_d}\right), \theta_2 = \arctan\left(\frac{\ell}{y_d}\right), \theta = \arctan\left(\frac{1}{k}\right). \quad (3)$$

Then, by (2), (3) and part (b) of the trigonometric preliminaries, we have

$$\left\{ \begin{array}{l} 0 < \theta_1 < \frac{\pi}{4}, 0 < \theta_2 < \frac{\pi}{4}, 0 < \theta \leq \frac{\pi}{4} \\ \text{and } 0 < \theta_1 + \theta_2 < \frac{\pi}{2}, \tan \theta_1 = \frac{1}{x_d}, \tan \theta_2 = \frac{\ell}{y_d}, \tan \theta = \frac{1}{k} \end{array} \right\} \quad (4)$$

From (4) and part (a) of trigonometric preliminaries, it follows that

$$\begin{aligned} \tan(\theta_1 + \theta_2) &= \frac{\frac{1}{x_d} + \frac{\ell}{y_d}}{1 - \frac{1}{x_d} \cdot \frac{\ell}{y_d}}; \\ \tan(\theta_1 + \theta_2) &= \frac{y_d + \ell x_d}{x_d y_d - \ell}; \\ \tan(\theta_1 + \theta_2) &= \frac{d \cdot (y_d + \ell x_d)}{d x_d y_d - d \ell}. \end{aligned} \quad (5)$$

By (5) and the expressions for  $x_d$  and  $y_d$  (see beginning of the proof) we get

$$\begin{aligned}
\tan(\theta_1 + \theta_2) &= \frac{d^2 + k\ell d + k\ell d + \ell^2 \cdot (k^2 + 1)}{[dk + \ell(k^2 + 1)](k\ell + d) - d\ell}; \\
\tan(\theta_1 + \theta_2) &= \frac{d^2 + 2k\ell d + \ell^2 \cdot (k^2 + 1)}{d\ell k^2 + k\ell^2(k^2 + 1) + kd^2 + \ell dk^2 + d\ell - d\ell}; \quad (6) \\
\tan(\theta_1 + \theta_2) &= \frac{d^2 + 2k\ell d + \ell^2 \cdot (k^2 + 1)}{k \cdot [2dk\ell + d^2 + \ell^2(k^2 + 1)]} = \frac{1}{k} = \tan \theta; \\
\tan(\theta_1 + \theta_2) &= \tan \theta
\end{aligned}$$

By (6) and part (b) of the trigonometric preliminaries, it follows that  $\theta_1 + \theta_2 = \theta$ , which combined with (3), clearly establishes that the pair  $(x_d, y_d)$  is a solution to (1).

Now, the converse. Suppose that  $(x, y)$  is a positive integer solution to (1).

Then

$$\left( 0 < \frac{1}{x} \leq 1, \quad 0 < \frac{\ell}{y} \leq \ell, \quad 0 < \frac{1}{k} \leq 1 \right) \quad (7)$$

Using (7), the trigonometric preliminaries, parts (a) and (b) and by taking tangent of both sides of (1), we obtain,

$$\frac{\frac{1}{x} + \frac{\ell}{y}}{1 - \frac{1}{x} \frac{\ell}{y}} = \frac{1}{k}$$

or equivalently

(Note that since  $0 < \frac{1}{k} \leq 1$ . The equal sides of (1) can be utmost equal to  $\frac{\pi}{4}$  )

$$\begin{aligned}
xy - \ell &= k(y + x\ell); \\
y \cdot (x - k) &= \ell \cdot (1 + kx)
\end{aligned} \quad (8)$$

Equation (8) shows that  $y$  is a divisor of the product  $\ell(1+kx)$ . But, by (1), we know that  $\gcd(\ell, y) = 1$ . Thus, by Result 1 (Euclid's lemma), it follows that  $y$  must divide  $1+kx$ ; and so,

$$\left\{ \begin{array}{l} 1+kx = y \cdot v \\ v \text{ a positive integer} \end{array} \right\} \quad (9)$$

By (9) and (8) we have that,

$$x = \ell \cdot v + k \quad (10)$$

From (9) and (10) we further get

$$1 + k(\ell v + k) = yv;$$

or equivalently

$$k^2 + 1 = (y - \ell k) \cdot v \quad (11)$$

Since  $v$  is a positive integer, equation (11) shows that  $(y - \ell k)$  is a positive integer divisor of  $k^2 + 1$ . Let  $y - \ell k = d$ ,  $d$  a positive divisor of  $k^2 + 1$ . Then  $y = \ell k + d$  and by (11) and (10) we also get

$$x = k + \ell \cdot \left( \frac{k^2 + 1}{d} \right),$$

which proves that the solution  $(x, y)$  has the required form.

Finally, we see by inspection that the  $N$  (number of positive divisors of  $k^2 + 1$ ) positive integer solutions to (1) are distinct since, obviously, all the  $Ny$ -coordinates are distinct. The proof is complete.

□

## 5 A listing of nine equalities

Let  $k$  and  $\ell$  be positive integers such that  $\gcd(\ell, k^2 + 1) = 1$ . Applying Theorem 1 with  $d = 1$  and  $d = k^2 + 1$  produces two inequalities.

1.  $\arctan\left(\frac{1}{k + \ell(k^2 + 1)}\right) + \arctan\left(\frac{\ell}{k\ell + 1}\right) = \arctan\left(\frac{1}{k}\right)$
2.  $\arctan\left(\frac{1}{k + \ell}\right) + \arctan\left(\frac{\ell}{k\ell + k^2 + 1}\right) = \arctan\left(\frac{1}{k}\right)$

Next, applying Theorem 1 with  $k = \ell = 1$ , produces the equality:

3.  $\arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{2}\right) = \frac{\pi}{4}$

For  $\ell = 1$  and  $k = 2$ :

4.  $\arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{7}\right) = \arctan\left(\frac{1}{2}\right)$ .

For  $\ell = 1$  and  $k = 3$

5.  $\arctan\left(\frac{1}{11}\right) + \arctan\left(\frac{1}{4}\right) = \arctan\left(\frac{1}{3}\right)$
6.  $\arctan\left(\frac{1}{8}\right) + \arctan\frac{1}{5} = \arctan\left(\frac{1}{3}\right)$

For  $\ell = 2$  and  $k = 4$ :

7.  $\arctan\left(\frac{1}{38}\right) + \arctan\left(\frac{2}{9}\right) = \arctan\left(\frac{1}{4}\right)$
8.  $\arctan\left(\frac{1}{6}\right) + \arctan\left(\frac{2}{25}\right) = \arctan\left(\frac{1}{4}\right)$

For  $\ell = 1$  and  $k = 6$ :

9.  $\arctan\left(\frac{1}{43}\right) + \arctan\left(\frac{1}{7}\right) = \arctan\left(\frac{1}{6}\right)$



## References

- [1] Unal, Hasan, *Proof without words: an arctangent equality*, Mathematics and Computer Education, Fall 2011, Vol. 45, No. 3, p 197.
- [2] Rosen, Kenneth H., *Elementary Number Theory and Its Applications*, 5th edition, Pearson, Addison Wesley, 2005.

For Result 1 (Lemma 3.4 in the above book), see page 109.

For Result 2 (Theorem 7.9 in the above book), see page 252.